ISRAEL JOURNAL OF MATHEMATICS **152** (2006), 83–104

CLASSIFICATION OF COCYCLES OF GENERIC EQUIVALENCE RELATIONS

BY

VYACHESLAV KULAGIN

Institute for Low Temperature Physics & Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., 61103 Kharkov, Ukraine
e-mail: kulagin@ilt.kharkov.ua

ABSTRACT

We study cocycles of an ergodic generic countable equivalence relation $\mathcal{R}$ modulo meager sets. Two cocycles of $\mathcal{R}$ are called weakly equivalent if they are cohomologous up to an element of $\operatorname{Aut}\mathcal{R}$. It is proved that two nontransient cocycles with values in an arbitrary countable group are weakly equivalent if and only if their generic Mackey actions are isomorphic.

0. Introduction

This paper proceeds with the study of cocycles in generic dynamics initiated in [5], [6]. Recall that a feature of the generic setting is that all subjects are studied modulo topologically negligible — meager — sets (see [19] and a survey [20]). We consider cocycles on a generic equivalence relation, i.e. generated by a countable homeomorphism group of a perfect Polish space, and investigate the problem of their classification under a weak equivalence. The notion of weak equivalence for cocycles was introduced originally in measurable dynamics (see a survey [1]). In the generic sense it means that the cocycles are cohomologous up to a homeomorphism that preserves the orbits ([5]). This orbit theoretical notion generalizes the concept of weak (or, equivalently, orbit) equivalence of dynamical systems.

The basics of generic orbit theory for countable group actions by homeomorphisms on a Polish space have been created in the celebrated work of Sullivan–Weiss–Wright ([19]). They proved that, modulo meager sets, there is only one,

Received February 27, 2004 and in revised form February 16, 2005

up to an isomorphism, equivalence relation generated by an ergodic countable homeomorphism group of a perfect Polish space ([19, Th. 1.8]). Thus it turned out that the situations in the orbit theory of generic and amenable measurable countable dynamical systems systems are very close (recall that the Connes–Feldman–Weiss theorem says that any two free ergodic measure-preserving actions of countable amenable groups on Lebesgue space are orbit equivalent [3]). At the same time the cocycle theory has been under an extensive research mainly in the measure theoretic setting (we refer the reader to the surveys [1], [17], [18], [21]). These facts motivated a parallel studying of the generic case, beginning from the work [5]. It was proved there that any two ergodic cocycles with values in an arbitrary Polish group are weakly equivalent. This uniqueness type theorem in generic dynamics works just as the well-known uniqueness theorem ([7], [9], [4]) for cocycles with dense ranges in measurable dynamics.

In the present paper we deal with cocycles taking values in countable groups. Such cocycles are classified up to weak equivalence in terms of so-called Mackey actions. The notion of generic Mackey action associated with a cocycle of a generic equivalence relation was introduced in [6] by analogy with its measurable version (see [15], [7]). Recall that it generalizes such concepts as the flow built under a function and (in the measurable case) Poincaré flow. It was shown in [6] that the generic Mackey action is an invariant of weak equivalence for Polish group valued cocycles. Our main Theorem 2.2 says that the generic Mackey action is a complete invariant of weak equivalence for nontransient cocycles with values in an arbitrary countable group. The same result for the simpler case of transient cocycles, i.e. those for which the skew product action is smooth, was proved in an earlier paper [6]. Thereby is produced a generic counterpart of the known classification result for cocycles of countable amenable group actions on Lebesgue space in terms of associated Mackey actions. The latter result in its most general form for cocycles with values in locally compact groups was established in a series of papers [2], [7], [8], [9] (starting, in fact, from Krieger's classification of hyperfinite type III equivalence relations [13]). One should note also the main distinction: the generic case assumes nothing like amenability for group actions in the measure theoretic setting.

Among other previous results we mention a classification of cocycles taking values in Polish groups with transitive generic Mackey actions [6]. The reader is also referred to [5] for applications of the generic cocycle theory to such problems as an outer conjugacy for homeomorphism groups and a classification of subrelations of generic equivalence relations.

1. Background and preliminaries

Throughout this paper X stands for a perfect Polish space. Recall that X is a Baire space, i.e. any open nonempty subset of X is nonmeager ([14]). Any G_δ-subset of X is again Polish in its relative topology ([14]).

The following proposition is to be applied below without mention.

PROPOSITION 1.1 ([14] or [5]): *Let Y be a Baire topological space and $A_n \subset Y$ ($n \in \mathbb{N}$) be a countable family of sets having the Baire property. Then there exists a dense G_δ-subset Z of Y such that $A_n|_Z$ is open in Z for all $n \in \mathbb{N}$.*

A Borel bijection Θ of X is called a **pseudo-homeomorphism** if $\Theta|_Y$ is a homeomorphism of Y for some dense G_δ-subset $Y \subset X$.

Two Baire spaces Ω_1 and Ω_2 are said to be **pseudo-homeomorphic** if there exist meager sets $M \subset \Omega_1$, $N \subset \Omega_2$ with $\Omega_1 \setminus M$ being homeomorphic to $\Omega_2 \setminus N$. It should be noted that any two perfect Polish spaces are pseudo-homeomorphic (see [19]).

For an equivalence relation $\mathcal{R}$ on X let $\mathcal{R}[A]$ denote the $\mathcal{R}$-saturation of $A \subset X$. If G is a group of bijections on X, let $\mathcal{R}_G = \{(x, gx) \colon x \in X, g \in G\}$. The $\mathcal{R}_G$-saturation of A will be denoted by $G[A]$. A set $T \subset X$ is called a **transversal** for $\mathcal{R}$ if it meets each $\mathcal{R}$-equivalence class at exactly one point. A Borel equivalence relation $\mathcal{R}$ is **smooth** if there exists a Borel map $f\colon X \to Y$, where Y is a standard Borel space, such that $(x, x') \in \mathcal{R} \Leftrightarrow f(x) = f(x')$.

Given two actions $U_1(G_1)$ and $U_2(G_2)$ of groups G_1, G_2 on X, denote by $\{U_1(G_1), U_2(G_2)\}$ the group of transformations of X generated by these two actions.

Let $\mathcal{R}$ be a countable Borel equivalence relation on X. $\mathcal{R}$ is called **generic** if it is generated by some countable pseudo-homeomorphism group Γ of X, i.e. $\mathcal{R} = \mathcal{R}_\Gamma$. Suppose that a countable Borel equivalence relation $\mathcal{R}$ on X satisfies the following condition: the saturation $\mathcal{R}[M]$ of any meager set $M \subset X$ is meager. It was shown in [19] that in this case $\mathcal{R}$ is generic. Moreover, there exists an $\mathcal{R}$-invariant dense G_δ-subset $Y \subset X$ such that $\mathcal{R}|_{Y \times Y}$ is generated by a countable homeomorphism group of Y ([19]).

Actions of countable groups G_1, G_2 by homeomorphisms of X are **generically orbit equivalent** if there exists a pseudo-homeomorphism Θ of X with $\Theta(\mathcal{R}_{G_1}[x]) = \mathcal{R}_{G_2}[\Theta x]$ for all x from some comeager subset of X. In other terms, the equivalence relations $\mathcal{R}_{G_1}$ and $\mathcal{R}_{G_2}$ are isomorphic.

Actions $(\Omega_1, W_1(G))$, $(\Omega_2, W_2(G))$ of a group G, with Ω_1, Ω_2 being Baire spaces, are **generically isomorphic** if there exists a homeomorphism Θ between invariant comeager subsets $\widetilde{\Omega}_1 \subset \Omega_1, \widetilde{\Omega}_2 \subset \Omega_2$ such that $\Theta W_1(g)\Theta^{-1}\omega =$

$W_2(g)\omega$ for all $g \in G$ and all $\omega \in \widetilde{\Omega}_2$.

From now on $\mathcal{R}$ stands for a generic countable equivalence relation on X. The set of its automorphisms $\operatorname{Aut}\mathcal{R}$ consists of pseudo-homeomorphisms Θ of X such that $\Theta(\mathcal{R}[x]) = \mathcal{R}[\Theta x]$ for all x from some comeager subset of X. The set $\operatorname{Int}\mathcal{R}$ of **inner** automorphisms of $\mathcal{R}$ is

$$\{\Theta \in \operatorname{Aut}\mathcal{R}\colon (\Theta x, x) \in \mathcal{R} \text{ for all } x \in X \text{ from some comeager subset of } X\}.$$

Suppose now that $\mathcal{R}$ is generated by a homeomorphism group Γ. We say that $\tau \in \operatorname{Int}\mathcal{R}$ is **strongly Γ-decomposable** on X if there exist a clopen partition $\{A_j\}$ $(j = 1, 2, \ldots)$ of X and a sequence $\{\gamma_j\} \subset \Gamma$ $(j = 1, 2, \ldots)$ such that $\tau x = \gamma_j x$ for $x \in A_j$.

A homeomorphism group G of a second countable Baire space Ω is called **ergodic** if there is $\omega_0 \in \Omega$ with $G\omega_0$ dense in Ω. Equivalently, any G-invariant set having the Baire property is either meager or comeager. Modulo meager sets, every orbit of an ergodic group G is dense (see [20], [6], [11]). An equivalence relation $\mathcal{R}_G$ on X is said to be ergodic if G is.

THEOREM 1.2 (Sullivan–Weiss–Wright, [19]): *Let G_1, G_2 be countable ergodic homeomorphism groups of a perfect Polish space X. Then, modulo a meager subset of X, $\mathcal{R}_{G_1}$ and $\mathcal{R}_{G_2}$ are isomorphic (or, in other terms, G_1 and G_2 are orbit equivalent).*

For $\mathcal{R}$ a generic equivalence relation let $\widetilde{\mathcal{R}}$ denote the equivalence relations on X which is called the **generic ergodic decomposition** ([20], [10]):

$$(x, y) \in \widetilde{\mathcal{R}} \Leftrightarrow \overline{\mathcal{R}[x]} = \overline{\mathcal{R}[y]}.$$

We recall some properties of $\widetilde{\mathcal{R}}$ (see also [6]). Every $\widetilde{\mathcal{R}}[x]$ is a G_δ-subset of X and, moreover, $\widetilde{\mathcal{R}}$ is a G_δ-subset of $X \times X$. If Γ is a homeomorphism group that generates $\mathcal{R}$ then the action of Γ on every $\widetilde{\mathcal{R}}[x]$ is minimal. This permits us to consider the equivalence class $\widetilde{\mathcal{R}}[x]$ as an ergodic component of x with respect to the Γ-action. Let Ω be the topological factor-space $X/\widetilde{\mathcal{R}}$, $\phi\colon X \to \Omega$ the projection. It has been shown in [6] that ϕ and Ω have the following properties: ϕ is an open map, Ω is a Baire, second countable T_0-space. For any meager $S \subset \Omega$, $\phi^{-1}(S)$ is meager too, and for any meager $\widetilde{\mathcal{R}}$-invariant $L \subset X$, $\phi(L)$ is meager.

PROPOSITION 1.3 ([6]): *Let $\mathcal{R}$ be a generic equivalence relation on a perfect Polish space X, $\widetilde{\mathcal{R}} \subset X \times X$ a generic ergodic decomposition of $\mathcal{R}$. Then there exists a dense $\widetilde{\mathcal{R}}$-invariant G_δ-set $X' \subset X$, a closed set T in X' that is* *transversal to $\widetilde{\mathcal{R}}$ on X', and an open continuous map $\pi\colon X' \to T$, where T is given the relative topology, such that $(x,y) \in \widetilde{\mathcal{R}} \Leftrightarrow \pi(x) = \pi(y)$.*

Thus, from the generic point of view, one may suppose that the factor-space $X/\widetilde{\mathcal{R}}$ of the ergodic decomposition is homeomorphic to T (so is Polish).

Let Γ be a countable homeomorphism group of X. Suppose that the $\widetilde{\mathcal{R}}_\Gamma$-transversal from 1.3 is a perfect space. Then X is said to be a **purely continuous** (resp. **discrete**) Γ-space if every $\widetilde{\mathcal{R}}_\Gamma$-orbit is a perfect (resp. discrete) space. By [6, 2.5], modulo meager sets, X always admits a decomposition as a union of clopen subsets $X = X_1 \cup X_2$ so that X_1 is a discrete Γ-space and X_2 is a purely continuous Γ-space. We note also that the discreteness of a Γ-space X is equivalent to having $\mathcal{R}_\Gamma$ is smooth modulo meager sets.

A Borel map $\alpha\colon \mathcal{R} \to G$ is called a **cocycle** of $\mathcal{R}$ with values in a Polish group G if for some $\mathcal{R}$-invariant dense G_δ-subset Y of X, $\alpha(x,y)\alpha(y,z) = \alpha(x,z)$ for all $(x,y),(y,z) \in \mathcal{R}|_{Y\times Y}$. The set of all cocycles of $\mathcal{R}$ with values in G is denoted by $Z^1(\mathcal{R},G)$ (with the identification of cocycles which differ only on a meager subset of X). Two cocycles $\alpha,\beta \in Z^1(\mathcal{R},G)$ are **cohomologous** ($\alpha \approx \beta$), if there exists a Borel map $f\colon X \to G$ such that $\alpha(x,y) = f(x)\beta(x,y)f(y)^{-1}$ for all $(x,y) \in \mathcal{R}$ modulo a meager subset of X.

Let θ be a homeomorphism of X without fixed points, $f\colon X \to G$ a Borel map. We will denote by $c(f) \in Z^1(\mathcal{R}_\mathbb{Z},G)$ a cocycle defined by the following: $c(f)(x,\theta^n x) = f(x)\cdot\ldots\cdot f(\theta^{n-1}x)$ if $n \geq 1$, $c(f)(x,\theta^n x) = e$ if $n = 0$, $c(f)(x,\theta^n x) = f(\Theta^{-1}x)^{-1}\cdot\ldots\cdot f(\Theta^n x)^{-1}$ if $n \leq -1$.

Given an $\alpha \in Z^1(\mathcal{R},G)$, a **skew product equivalence relation** $\mathcal{E}_\alpha$ on $X\times G$ is defined by $((x,g_1),(y,g_2)) \in \mathcal{E}_\alpha$ iff $(x,y) \in \mathcal{R}$ and $\alpha(y,x) = g_2{g_1}^{-1}$. Suppose that $\mathcal{R}$ is generated by a homeomorphism group Γ of X. Then $\mathcal{E}_\alpha$ is generated by a **skew product action** of Γ on $X \times G$: $\gamma(\alpha)(x,g) = (\gamma x, \alpha(\gamma x, x)g)$ ([5]). We denote this action by $\Gamma(\alpha)$. Modulo meager subsets of X, the skew product action is an action by homeomorphisms of $X \times G$ ([5]). A cocycle $\alpha \in Z^1(\mathcal{R},G)$ is called

1. **ergodic** if the skew product $\mathcal{E}_\alpha$ is ergodic;
2. **regular** if it is cohomologous to some ergodic cocycle with values in a closed subgroup H of G;
3. **transient** if $\Gamma(\alpha)$ is a discrete action ([6]).

Two cocycles $\alpha,\beta \in Z^1(\mathcal{R},G)$ are called **weakly equivalent** if there exists $\Theta \in \operatorname{Aut}\mathcal{R}$ with $\alpha \approx \beta\circ(\Theta\times\Theta)$. Obviously, each class from the above 1–3 is invariant under weak equivalence.

Let $V(G)$ denote the action of G on $X\times G$ defined by $V(g)(x,h) = (x,hg^{-1})$.

Clearly, this action commutes with the skew product action $\Gamma(\alpha)$. Let $\widetilde{\mathcal{E}}_\alpha$ be the generic ergodic decomposition equivalence relation of $\mathcal{E}_\alpha$. Let $\Omega = (X \times G)/\widetilde{\mathcal{E}}_\alpha$ be the topological factor-space and $\phi\colon X \times G \to \Omega$ be the factor map.

Definition 1.4: The action $W_\alpha(G)$ of the group G on the space Ω defined by

$$W_\alpha(g)\omega = \phi(V(g)y),$$

where $y \in \phi^{-1}(\omega)$, $\omega \in \Omega, g \in G$, is called the **generic Mackey action** associated with the cocycle α.

Let us recall the main properties of this action (see [6]). $W_\alpha(G)$ is a continuous action. It is ergodic iff $\mathcal{R}$ is. It is easy to check that for weakly equivalent cocycles $\alpha, \beta \in Z^1(\mathcal{R}, G)$ the corresponding generic Mackey actions are generically isomorphic ([6, 3.1]). Thus this is an invariant of weak equivalence for cocycles.

It follows from 1.3 (see also [6]) that for G countable one may consider the generic Mackey action as an action on a Polish space. For a class of transient cocycles with values in a countable group the generic Mackey action is a complete invariant of weak equivalence ([6, Th. 22]). We note also that for any ergodic countable group action $W(G)$ on a perfect Polish space there exists a cocycle for which the associated generic Mackey action is isomorphic to $W(G)$ (the existence theorem). In the case when $W(G)$ is free one may choose such a cocycle to be either of transient or nontransient type, and when $W(G)$ is nonfree such a cocycle is necessarily nontransient ([6]).

It follows from the definition that the generic Mackey action associated with an ergodic cocycle is trivial. The following uniqueness theorem provides the classification of ergodic cocycles [5] (cf. the uniqueness theorem for cocycles in measurable dynamics [7]):

THEOREM 1.5: *Suppose the cocycles $\alpha, \beta \in Z^1(\mathcal{R}, G)$, where G is an arbitrary Polish group, are ergodic. Then they are weakly equivalent.*

It is also shown in [6] that regularity of a cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is equivalent to transitivity of the generic Mackey action $W_\alpha(G)$. In this case $W_\alpha(G)$ is a natural action of G on G/H, where H is a closed subgroup of G, and it is a complete invariant of weak equivalence for regular cocycles with values in an arbitrary Polish group G.

The following proposition is an analogue of the topological Fubini theorem for open maps. We will use it and its simple corollary in our subsequent considerations mainly in an implicit form.

PROPOSITION 1.6 ([16]): *Suppose Z, S are Polish spaces, $p\colon Z \to S$ is a continuous open surjection, and A is a Baire subset of Z. Then A is meager if and only if $\{s \in S\colon A \cap p^{-1}(s)$ is nonmeager in $p^{-1}(s)\}$ is meager in S.*

COROLLARY 1.7: *Let Θ be a pseudo-homeomorphism of a perfect Polish space Z and S a closed transversal of a generic ergodic decomposition $\widetilde{\mathcal{R}}$ on Z. Suppose that $\widetilde{\mathcal{R}}[s]$ is Θ-invariant for every $s \in S$. Then there exists a meager $\mathcal{R}$- and Θ-invariant F_σ-subset M such that the following is true on $Z \setminus M$: $\Theta|_{\widetilde{\mathcal{R}}[s]}$ is a homeomorphism of $\widetilde{\mathcal{R}}[s]$ for each $s \in S$.*

2. A classification of cocycles

In this section we prove our main result on classification of cocycles (Theorem 2.2). We remind the reader that it is implicit that everything is considered modulo meager sets.

Throughout this section G will stand for a countable group, $\mathcal{R} = \mathcal{R}_\Gamma$ an ergodic generic countable equivalence relation generated by a countable homeomorphism group Γ. For $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$, we denote by Ω_α the factor-space $(X \times G)/\widetilde{\mathcal{E}}_\alpha$.

Let

$$\mathcal{R}(\Gamma, G) = \{((x,g),(x',g'))\colon (x,x') \in \mathcal{R}_\Gamma, g, g' \in G\}.$$

Then $\mathcal{R}_{\{\Gamma(\alpha),V(G)\}} = \mathcal{R}(\Gamma, G)$ for any $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$. Given any $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ we define a cocycle $\varphi_\alpha \in Z^1(\mathcal{R}_{\{\Gamma(\alpha),V(G)\}}, G)$ by

$$\varphi_\alpha(V(g)\gamma(\alpha)(x,\overline{g}),(x,\overline{g})) = g.$$

It is well-defined because the relation $V(g_1^{-1}g_2)(\gamma_1^{-1}\gamma_2)(\alpha)(x,g) = (x,g)$ implies $\gamma_1^{-1}\gamma_2 x = x$. Thus $\alpha(\gamma_1^{-1}\gamma_2 x, x) = e$ and hence $g_1 = g_2$.

First, we state the following auxiliary proposition:

PROPOSITION 2.1: *Two cocycles $\alpha, \beta \in Z^1(\mathcal{R}_\Gamma, G)$ are weakly equivalent if and only if the cocycles φ_α and φ_β are weakly equivalent.*

Proof: Suppose $\alpha(\Phi x, \Phi x') = \zeta(x)\beta(x,x')\zeta(x')^{-1}$ for all $(x,x') \in \mathcal{R}_\Gamma$, where $\Phi \in \operatorname{Aut}\mathcal{R}_\Gamma$, $\zeta\colon X \to G$ is a Borel map. Define a map $\Theta\colon X \times G \to X \times G$ by $\Theta(x,g) = (\Phi x, \zeta(x)g)$. Then it is routine to verify that $\Theta \in \operatorname{Aut}\mathcal{R}(\Gamma, G)$ and $\varphi_\beta \circ (\Theta \times \Theta) = \varphi_\alpha$.

Suppose $\varphi_\alpha(\Theta z, \Theta z') = \zeta(z)^{-1}\varphi_\beta(z,z')\zeta(z')$ for all $(z,z') \in \mathcal{R}(\Gamma, G)$, where $\Theta \in \operatorname{Aut}\mathcal{R}(\Gamma, G)$, $\zeta\colon X \times G \to G$ is a Borel map. Let $vx = (\pi_X \circ \Theta)(x,e)$, where π_X denotes the projection $X \times G \to X$. Then $v\colon X \to X$ is a continuous map with $v^{-1}(x)$ at most countable for any $x \in X$. Let U be a proper clopen subset of X with $Y = v^{-1}(U)$ being a proper subset of X. Let

$$\mathcal{R}_v = \{(y_1, y_2) \in Y \times Y \colon vy_1 = vy_2\}.$$

Then $\mathcal{R}_v$ is a smooth generic subrelation of $\mathcal{R}_\Gamma|_{Y\times Y}$ and hence it is of discrete type ([6]). This implies that there exists a clopen $\mathcal{R}_v$-transversal B with $v\colon B \to U$ a homeomorphism ([6]). By virtue of [19, 1.6] there exist homeomorphisms δ, $\delta' \in \operatorname{Int}\mathcal{R}_\Gamma$ with $\delta = \delta^{-1}$, $\delta' = \delta'^{-1}$, $\delta B = X \setminus B$, $\delta' v(B) = X \setminus v(B)$. Define a homeomorphism $\Phi\colon X \to X$ by setting: $\Phi x = vx$ for $x \in B$, $\Phi x = (\delta' \circ v \circ \delta)x$ for $x \in X \setminus B$. Then $\Phi \in \operatorname{Aut}\mathcal{R}_\Gamma$. It is easy to see that $(\Phi \times \mathrm{id})\Theta^{-1} \in \operatorname{Int}\mathcal{R}(\Gamma, G)$ so $\Theta = \tau(\Phi \times \mathrm{id})$ for some $\tau \in \operatorname{Int}\mathcal{R}(\Gamma, G)$. Hence for $(x, x') \in \mathcal{R}_\Gamma$ one has

$$\begin{aligned}
&\varphi_\alpha(\Theta(x,e), \Theta(x',e)) = \varphi_\alpha(\tau(\Phi \times \mathrm{id})(x,e), (\tau(\Phi \times \mathrm{id})(x',e)) \\
=&\varphi_\alpha(\tau(\Phi x, e), (\Phi x, e))\varphi_\alpha((\Phi x, e), (\Phi x', e))\varphi_\alpha((\Phi x', e), \tau(\Phi x', e)) \\
=&\xi(x)\alpha(\Phi x, \Phi x')\xi(x')^{-1}
\end{aligned}$$

where $\xi(x) = \varphi_\alpha(\tau(\Phi x, e), (\Phi x, e))$, so $\xi\colon X \to G$ is a Borel map.

On the other hand

$$\begin{aligned}
\varphi_\alpha(\Theta(x,e), \Theta(x',e)) &= \zeta(x,e)^{-1}\varphi_\beta((x,e)(x',e))\zeta(x',e) \\
&= \zeta(x,e)^{-1}\beta(x,x')\zeta(x',e).
\end{aligned}$$

Thus

$$\xi(x)^{-1}\zeta(x,e)^{-1}\beta(x,x')\zeta(x',e)\xi(x') = \alpha(\Phi x, \Phi x')$$

for all $(x, x') \in \mathcal{R}_\Gamma$. ∎

THEOREM 2.2: *Let $\mathcal{R}$ be an ergodic generic countable equivalence relation on a Polish space X and G a countable group. Two nontransient cocycles α, $\beta \in Z^1(\mathcal{R}_\Gamma, G)$ are weakly equivalent if and only if the generic Mackey actions $W_\alpha(G)$, $W_\beta(G)$ are (generically) isomorphic.*

Proof: If α is weakly equivalent to β then $W_\alpha(G)$ is generically isomorphic $W_\beta(G)$ by [6, 3.1].

Suppose now that the generic Mackey actions $W_\alpha(G)$, $W_\beta(G)$ are isomorphic. One may assume that Ω_α is homeomorphic to Ω_β and is Polish (by the countability of G and 1.3, see also [6]). It follows from the ergodicity of $W_\alpha(G)$ that Ω_α is either perfect or countable discrete space. In the second case $W_\alpha(G)$ is a transitive action and the cocycles α, β are weakly equivalent by [6, 3.7,3.8]. So we assume in the sequel that Ω_α is a perfect Polish space. Since α, β are nontransient cocycles, the skew-product actions $\Gamma(\alpha)$, $\Gamma(\beta)$ are both of purely continuous type ([6]).

Let $Z = X \times G$ and we hold this notation in all cases of discarding a meager subset from $X \times G$.

By virtue of [6, 2.4] there exists a closed transversal S_α of the equivalence relations $\widetilde{\mathcal{E}}_\alpha$ such that S_α is homeomorphic to Ω_α, the projection $\phi_\alpha\colon Z \to S_\alpha$ is open and continuous. Let S_β be a $\widetilde{\mathcal{E}}_\beta$-transversal with similar properties. One may think that $W_\alpha(G)$ is an action on S_α, $W_\beta(G)$ on S_β. Let $\rho\colon S_\beta \to S_\alpha$ denote the homeomorphism realizing the isomorphism of the generic Mackey actions $W_\alpha(G)$ and $W_\beta(G)$, i.e. $\rho W_\beta(G)\rho^{-1} = W_\alpha(G)$.

We suppose also that for every $g \in G$ the set $\{s \in S\colon W_\alpha(g)s = s\}$ is clopen in S_α (see 1.1).

It will be convenient for a while to distinguish the spaces on which φ_α and φ_β are defined. Denote them by Z_α and Z_β (so that both Z_α, Z_β are pseudo-homeomorphic to $X \times G$).

Let $H_s = \{g \in G\colon W_\alpha(g)s = s\}$ denote the stability group at $s \in S_\alpha$. It should be noted that the family $\{H_s\}_{s\in S_\alpha}$ has the following property (modulo meager subsets of S_α): for every $s \in S_\alpha$ and every $g \in H_s$ there exists a clopen neighborhood O of s with $g \in H_{\widehat{s}}$ for all $\widehat{s} \in O$.

It is easy to see that $\widetilde{\mathcal{E}}_\alpha[z]$ is a $V(H_{\phi(z)})$-invariant (closed) set for every $z \in Z_\alpha$. We denote by $\mathcal{R}^s_\alpha \subset \widetilde{\mathcal{E}}_\alpha[s] \times \widetilde{\mathcal{E}}_\alpha[s]$ an equivalence relation on $\widetilde{\mathcal{E}}_\alpha[s]$ generated by the two actions: $\Gamma(\alpha)$ and $V(H_s)$. Define a cocycle $\lambda^s_\alpha \in Z^1(\mathcal{R}^s_\alpha, H_s)$ by

$$\lambda^s_\alpha = \varphi_\alpha|_{\mathcal{R}^s_\alpha}.$$

LEMMA 2.3: *λ^s_α is an ergodic cocycle.*

Proof: It easily follows from the ergodicity of the action $\Gamma(\alpha)$ on $\widetilde{\mathcal{E}}_\alpha[s]$. ∎

The following property of the family of cocycles $\{\lambda^s_\alpha\}_{s\in S_\alpha}$ will be used in the sequel without mention: for every $s \in S_\alpha$, $g \in H_s$ there exists a clopen neighborhood O of s with $\lambda^{\widehat{s}}_\alpha(V(g)\gamma(\alpha)t, t) = g$ for all $\widehat{s} \in O$, $t \in \widetilde{\mathcal{E}}_\alpha[\widehat{s}]$, $\gamma \in \Gamma$.

Define a relation $\mathcal{H}_\alpha$ on Z_α by

$$\mathcal{H}_\alpha = \bigcup_{s\in S_\alpha} \mathcal{R}^s_\alpha$$

Note that $\mathcal{H}_\alpha = \mathcal{R}_{\{\Gamma(\alpha),V(G)\}} \cap \widetilde{\mathcal{E}}_\alpha$, so $\mathcal{H}_\alpha$ is a generic (Borel) equivalence subrelation of $\mathcal{R}_{\{\Gamma(\alpha),V(G)\}}$. Besides, the generic ergodic decomposition $\widetilde{\mathcal{H}}_\alpha$ of $\mathcal{H}_\alpha$ coincides with $\widetilde{\mathcal{E}}_\alpha$.

Let H_α be a countable pseudo-homeomorphism group of Z_α generating $\mathcal{H}_\alpha$ (thus $H_\alpha \subset \operatorname{Int}\mathcal{R}_{\{\Gamma(\alpha),V(G)\}}$). By virtue of 1.6, one may assume that H_α is a homeomorphism group of Z_α.

Define a cocycle $\lambda_\alpha \in Z^1(\mathcal{H}_\alpha, G)$ by

$$\lambda_\alpha = \varphi_\alpha|_{\mathcal{H}_\alpha}.$$

The same objects $\{\mathcal{R}_s^\beta\}_{s\in S_\beta}$, $\mathcal{H}_\beta$, $\widetilde{\mathcal{H}}_\beta$, H_β, $\{\lambda_\beta^s\}_{s\in S_\beta}$, λ_β for the cocycle β are to be constructed in a similar way.

By Proposition 2.1 it suffices to prove a weak equivalence of φ_α and φ_β. For this, we first show that the cocycles λ_α and λ_β are weakly equivalent.

PROPOSITION 2.4: *There exists a pseudo-homeomorphism Θ: $Z_\beta \to Z_\alpha$ such that $(\Theta\times\Theta)\mathcal{H}_\alpha = \mathcal{H}_\beta$ and*

$$\lambda_\alpha(z_1, z_2) = \lambda_\beta(\Theta z_1, \Theta z_2)$$

for all $(z_1, z_2) \in \mathcal{H}_\alpha$ modulo a meager subset of Z_α.

We need some preliminary results of a technical nature.

LEMMA 2.5: *Let Γ be a countable group and Z a purely continuous (non-ergodic) perfect Polish Γ-space. Let S be a closed $\widetilde{\mathcal{R}}_\Gamma$-transversal on Z, $\{s_k\}_{k=1}^\infty$ a countable dense subset of S. Suppose that M is a meager F_σ-subset of Z with $\mathcal{R}_\Gamma[M] \cap \{s_k\}_{k=1}^\infty = \emptyset$. Then there exists a dense $\mathcal{R}_\Gamma$-invariant G_δ-set $Y \subset Z$ with $M \subset Z\setminus Y$, $Y \supset \{s_k\}_{k=1}^\infty$, Y being a purely continuous Γ-space, $S\cap Y$ a transversal for $\widetilde{\mathcal{R}}_\Gamma|_{Y\times Y}$.*

Proof: One easily checks that $\mathcal{R}_\Gamma[M]\cap S$ is meager in S. Let

$$M_1 = \widetilde{\mathcal{R}}_\Gamma[\mathcal{R}_\Gamma[M]\cap S], \quad M_2 = \mathcal{R}_\Gamma[M],$$

so that M_1, M_2 are meager F_σ-subsets in Z. Then the set $Y' = X\setminus(M_1\cup M_2)$ is a $\mathcal{R}_\Gamma$-invariant G_δ-subset of Z containing $\{s_k\}_{k=1}^\infty$, and $S\cap Y'$ is a $\widetilde{\mathcal{R}}_\Gamma$-transversal in Y' (recall that $\widetilde{\mathcal{R}}_\Gamma|_{Y'\times Y'} = \widetilde{\mathcal{R}}_{\Gamma|_{Y'}}$). Let D be the set of points from $S\cap Y'$ with discrete $\widetilde{\mathcal{R}}_{\Gamma|_{Y'}}$-orbits. Then the set of all points from Y' with discrete $\widetilde{\mathcal{R}}_{\Gamma|_{Y'}}$-orbits is $\widetilde{D} = \widetilde{\mathcal{R}}_{\Gamma|_{Y'}}[D]$. Note that $D\cap\{s_k\}_{k=1}^\infty = \emptyset$. Indeed, if $\widetilde{\mathcal{R}}_{\Gamma|_{Y'}}[s_k]$ is discrete then the F_σ-set $\mathcal{R}_\Gamma[M]\cap\widetilde{\mathcal{R}}_\Gamma[s_k]$ is nonmeager in $\widetilde{\mathcal{R}}_\Gamma[s_k]$. So it has a nonempty interior in $\widetilde{\mathcal{R}}_\Gamma[s_k]$ and $\gamma s_k \in \mathcal{R}_\Gamma[M]$ for some $\gamma\in\Gamma$ that contradicts the assumption of the lemma. By [6, 2.6] D is F_σ in $S\cap Y'$, which implies $\widetilde{D}$ is a meager F_σ-subset of Y'. Put $Y = Y'\setminus\widetilde{D}$ to complete the proof. ∎

LEMMA 2.6 (cf. [19, 1.5]): *Let F, H be countable homeomorphism groups of a perfect Polish space Z. Suppose that every $h \in H$ is strongly F-decomposable on Z and the set $\{z: fz = z\}$ is clopen for every $f \in F$. Let T be a countable dense F- and H-invariant subset of Z, and suppose that for every $t \in T$ and every $f \in F$ there exists $h \in H$ with $ft = ht$. Then there exists a dense F- and H-invariant G_δ-set $Y \subset Z$ containing T, on which F and H are strongly orbit equivalent.*

Proof: See [19, 1.5] for a suitable idea. ∎

Remark: When Z is a purely continuous F-space and $T = F[S^d]$, where S^d is a countable dense subset of a closed $\widetilde{\mathfrak{R}}_F$-transversal S, one may choose Y to be a purely continuous space again, and $S \cap Y$ a $\widetilde{\mathfrak{R}}_F$-transversal on Y (see 2.5).

LEMMA 2.7: *Let Γ be a countable group and Z a purely continuous (nonergodic) perfect Polish Γ-space. Let S be a closed $\widetilde{\mathfrak{R}}_\Gamma$-transversal on Z such that the projection ϕ: $Z \to S$ is open and continuous. Then given any $s \in S$ there exists a basis of neighborhoods $\mathcal{B}$ of s in Z consisting of clopen sets, such that $\widetilde{\mathfrak{R}}_\Gamma[B] \cap S = B \cap S$, $\widetilde{\mathfrak{R}}_\Gamma[B]$ is clopen in Z for each $B \in \mathcal{B}$.*

Proof: Given an arbitrary clopen neighborhood A of s the set $\widetilde{\mathfrak{R}}_\Gamma[A \cap S]$ is clopen in Z. Put $B = A \cap \widetilde{\mathfrak{R}}_\Gamma[A \cap S]$. It is easy to see that $\widetilde{\mathfrak{R}}_\Gamma[B] \cap S = B \cap S$. Besides, $\widetilde{\mathfrak{R}}_\Gamma[B] = \widetilde{\mathfrak{R}}_\Gamma[\widetilde{\mathfrak{R}}_\Gamma[B] \cap S] = \widetilde{\mathfrak{R}}_\Gamma[B \cap S]$, so $\widetilde{\mathfrak{R}}_\Gamma[B]$ is clopen. Now the statement of the lemma follows from the fact that the family of clopen neighborhoods of s forms a basis. ∎

In the sequel the elements of $\{0,1\}^n$ will be denoted by $\underline{\mu} = (\mu_1, \ldots, \mu_n)$. We also fix a metric on Z_α. Let $B(z, \varepsilon)$ denote a ball centered in $z \in Z_\alpha$ with the radius ε, and let $\operatorname{diam}(D)$ denote the diameter of a closed subset $D \subset Z_\alpha$.

LEMMA 2.8: *Let $S_\alpha^d = \{s_k\}_{k=1}^\infty$ be a countable dense subset of S_α. Let $\widehat{s}_k = \rho^{-1}(s_k)$, $S_\beta^d = \{\widehat{s}_k\}_{k=1}^\infty$. Let $\Upsilon = H_\alpha[S_\alpha^d] \setminus S_\alpha^d = \{\tau_i\}_{i=1}^\infty$, $\widehat{\Upsilon} = H_\beta[S_\beta^d] \setminus S_\beta^d = \{\widehat{\tau}_i\}_{i=1}^\infty$. Then the following conditions are satisfied simultaneously:*

1. *There exist a sequence of pairwise commuting homeomorphisms $\{f_n\}_{n=1}^\infty \subset \operatorname{Int} \mathcal{H}_\alpha$ with $f_n = {f_n}^{-1}$ for all $n \in \mathbb{N}$.*
2. *There exists a refining sequence of clopen partitions of Z_α: $\{\{K^n(\underline{\mu}) : \underline{\mu} \in \{0,1\}^n\}\}_{n=1}^\infty$ such that $K^n(\underline{\mu}) = K^{n+1}(\underline{\mu}, 0) \cup K^{n+1}(\underline{\mu}, 1)$, $\underline{\mu} \in \{0,1\}^n$.*
3. *$f_1^{\mu_1} \cdot \ldots \cdot f_n^{\mu_n} K^n(\underline{0}) = K^n(\underline{\mu})$, $\underline{\mu} \in \{0,1\}^n$.*
4. *$S_\alpha \subset K^n(\underline{0})$ for all $n \in \mathbb{N}$.*

5. $K^n(\underline{0}) = \bigcup_{q=1}^{\infty} L_{n,q}$, *where each* $L_{n,q}$ *is a clopen subset of* Z_α *with* $\operatorname{diam}(L_{n,q}) < 1/n$, $L_{n,q} \cap S_\alpha = \widetilde{\mathcal{H}}_\alpha[L_{n,q}] \cap S_\alpha$, $L_{n,q} \cap L_{n,p} = \emptyset$ $(q \neq p)$.
6. $\{\tau_1, \dots, \tau_n\} \subset \{f_1^{\mu_1} \dots f_{2n}^{\mu_{2n}} S_\alpha^d \colon \underline{\mu} \in \{0,1\}^{2n}\}$.
7. *There exist similar objects on* Z_β: $\{\widehat{f}_n\}_{n=1}^{\infty}$, $\{\widehat{K}^n(\underline{\mu})\colon \underline{\mu} \in \{0,1\}^n\}_{n=1}^{\infty}$, $\{\{\widehat{L}_{n,q}\}_{q=1}^{\infty}\}_{n=1}^{\infty}$ *associated to* β *such that the conditions analogous to 1–6 are true for them.*
8. $\lambda_\alpha(f_1^{\mu_1} \dots f_n^{\mu_n} z, z) = \lambda_\beta(\widehat{f}_1^{\mu_1} \dots \widehat{f}_n^{\mu_n} \widehat{z}, \widehat{z})$ *for all* $\underline{\mu} \in \{0,1\}^n$, $n \in \mathbb{N}$, *and all pairs* $(z, \widehat{z}) \in K^n(\underline{0}) \times \widehat{K}^n(\underline{0})$ *with* $\phi_\alpha(z) = \rho(\phi_\beta(\widehat{z}))$.

Proof: We begin with the following lemma.

LEMMA 2.9: *Under the assumptions of Lemma 2.8, suppose* A, B *are clopen disjoint subsets of* Z_α *such that* $\phi_\alpha(A) = \phi_\alpha(B)$, $\xi\colon S_\alpha \to G$ *is a continuous map. Then there exist a dense* $\mathcal{H}_\alpha$*-invariant* G_δ*-subset* $Z_1 \subset Z_\alpha$ *which is a* H_α*-space of purely continuous type,* $Z_1 \supset S_\alpha^d$, *and a homeomorphism* δ *of* Z_1 *with the following properties on* Z_1: $\delta \in \operatorname{Int}\mathcal{H}_\alpha$, $\delta = \delta^{-1}$, $\delta A = B$, $\delta = \mathrm{id}$ *outside of* $A \cup B$, *and for every* $s \in S_\alpha$, $\lambda_\alpha^s(\delta z, z) = \xi(s)$ *for all* $z \in \widetilde{\mathcal{H}}_\alpha[s] \cap A$.

Proof: Let $S_A = \mathcal{H}_\alpha[S_\alpha^d] \cap A = \{s_j'\}_{j=1}^{\infty}$, $S_B = \mathcal{H}_\alpha[S_\alpha^d] \cap B = \{s_k''\}_{k=1}^{\infty}$. Let $Q = \phi_\alpha(A) = \phi_\alpha(B)$. Recall that given any $a \in A$, $g \in G$, there exists $\widetilde{h}_a \in H_\alpha$ such that $\widetilde{h}_a a \in B$ and $\lambda_\alpha^{\phi_\alpha(a)}(\widetilde{h}a, a) = g$ (the ergodicity of λ_α^s).

Choose $\widetilde{h}_1 \in H_\alpha$ with $\widetilde{h}_1 s_1' \in B$ and $\lambda_\alpha^{\phi_\alpha(s_1')}(\widetilde{h}_1 s_1', s_1') = \xi(\phi_\alpha(s_1'))$. We claim that there is a clopen neighborhood U_1 of s_1' with $U_1 \subsetneqq A$, $\widetilde{h}_1 U_1 \subsetneqq B$, $\lambda_\alpha(\widetilde{h}_1 u, u) = \xi(\phi_\alpha(s_1'))$ for all $u \in U_1$, and such that the following condition is satisfied:

$$(I)\qquad \begin{array}{l} \text{For each } a \in A \setminus U_1 \text{ there is } \widetilde{h}' \in H_\alpha \text{ with } \widetilde{h}'a \in B \setminus \widetilde{h}_1 U_1, \\ \text{and for each } b \in B \setminus \widetilde{h}_1 U_1 \text{ there is } \widetilde{h}'' \in H_\alpha \text{ with } \widetilde{h}''b \in A \setminus U_1. \end{array}$$

Indeed, the set $H_\alpha[s_1'] \cap A$ is infinite, so there exists $\overline{h}_1 \in H_\alpha : \overline{h}_1 s_1' \in A$ and $\overline{h}_1 s_1' \neq s_1'$. Let V_1 be a neighborhood of s_1', with $V_1 \subsetneqq A$ and $V_1 \cap \overline{h}_1 V_1 = \emptyset$. Similarly there exists $\overline{h}_2 \in H_\alpha$: $\overline{h}_2 \widetilde{h}_1 s_1' \in B$ and $\overline{h}_2 \widetilde{h}_1 s_1' \neq \widetilde{h}_1 s_1'$. Let V_2 be a neighborhood of $\widetilde{h}_1 s_1'$ with $V_2 \subsetneqq B$ and $V_2 \cap \overline{h}_2 V_2 = \emptyset$. Now choose a clopen U_1 such that $U_1 \subset V_1$ and $\widetilde{h}_1 U_1 \subset V_2$.

Let s_{k_2}'' be the first term in the enumeration of S_B that belongs to $B \setminus \widetilde{h}_1 U_1$. The ergodicity of λ_α^s $(s \in S_\alpha)$ and the condition (I) imply that there exists $\widetilde{h}_2^{-1} \in H_\alpha$: $\widetilde{h}_2^{-1} s_{k_2}'' \in A \setminus U_1$. Find a clopen neighborhood U_2 of $\widetilde{h}_2^{-1} s_{k_2}''$ with $U_2 \subsetneqq A \setminus U_1$, $\widetilde{h}_2 U_2 \subsetneqq B \setminus \widetilde{h}_1 U_1$, $\lambda_\alpha(\widetilde{h}_2 u, u) = \xi(\phi_\alpha(s_{k_2}''))$ for $u \in U_2$, and such that the condition similar to (I) holds: for each $a \in A \setminus (U_1 \cup U_2)$ there is $\widetilde{h}' \in H_\alpha$:

$\widetilde{h}'a \in B\setminus(\widetilde{h}_1U_1\cup\widetilde{h}_2U_2)$, and for each $b \in B\setminus(\widetilde{h}_1U_1\cup\widetilde{h}_2U_2)$ there exists $\widetilde{h}'' \in H_\alpha$: $\widetilde{h}''b \in A\setminus(U_1\cup U_2)$. Now proceed by induction. For m odd find the first term s'_{j_m} in the enumeration of S_A such that $s'_{j_m} \in A\setminus(U_1\cup\cdots\cup U_{m-1})$ together with a clopen neighborhood U_m of s'_{j_m} and $\widetilde{h}_m \in H_\alpha$ which satisfy a condition similar to (I) and the condition $\lambda_\alpha(\widetilde{h}_mu,u) = \xi(\phi_\alpha(s'_{j_m}))$ for all $u \in U_m$. If m is even, find $s''_{k_{m+1}}$, $\widetilde{h}_{m+1}$ and U_{m+1} with the same properties. As a result we obtain $S_A \subset \bigcup_{m=1}^\infty U_m \subset A$, $S_B \subset \bigcup_{m=1}^\infty \widetilde{h}_mU_m \subset B$. Put $Z_2 = \bigcup_{m=1}^\infty(U_m\cup\widetilde{h}_mU_m)$. Then the set $F = (A\cup B)\setminus Z_2$ is closed meager in Z_α and $\mathcal{H}_\alpha[F]\cap S_\alpha^d = \emptyset$. It remains to apply 2.5 to find a desired set Z_1 on which a homeomorphism δ is defined by: $\delta z = \widetilde{h}_mz$ if $z \in U_m\cap Z_1$, $\delta z = \widetilde{h}_m^{-1}z$ if $z \in \widetilde{h}_mU_m\cap Z_1$, $\delta z = z$ if $z \in Z_1\setminus(A\cup B)$. ∎

Return to the *Proof of Lemma 2.8*: Let $\mathcal{R}_\alpha^{s_k}[s_k] = \{s_k^j\}_{j=1}^\infty$, where $s_k^1 = s_k$. Let $B_k^j = B(s_k, 1/j)$. Proceed by induction on n. Let $\tau_1 = s_1^2 = \widetilde{h}s_1$ for some $\widetilde{h} \in H_\alpha$. Apply 2.7 to find a clopen neighborhood $A(s_1)$ of s_1 with

$$A(s_1) \subset B_1^1, \quad \widetilde{\mathcal{H}}_\alpha[A(s_1)]\cap S_\alpha = A(s_1)\cap S_\alpha, \quad A(s_1)\cap\widetilde{h}A(s_1) = \emptyset,$$

and $\widetilde{\mathcal{H}}_\alpha[A(s_1)]$ being clopen in Z_α. Let $B = \widetilde{\mathcal{H}}_\alpha[A(s_1)]\setminus A(s_1)$. It is easy to see that $\phi_\alpha(B) = \phi_\alpha(A(s_1))$. By virtue of Lemma 2.9, there exists a homeomorphism $\delta_{1,1} \in \operatorname{Int}\mathcal{H}_\alpha$ which satisfies: $\delta_{1,1} = {\delta_{1,1}}^{-1}$, $\delta_{1,1}A(s_1) = B$, $\delta_{1,1}s_1 = \tau_1$, $\delta_{1,1} = \mathrm{id}$ outside of $\widetilde{\mathcal{H}}_\alpha[A(s_1)]$, for each $s \in A(s_1)\cap S_\alpha$

$$\lambda_\alpha(\delta_{1,1}z, z) = \xi_{1,1}(s)$$

for all $z \in A(s_1)\cap\widetilde{\mathcal{H}}_\alpha[s]$, where $\xi_{1,1}\colon \phi_\alpha[A(s_1)]\cap S_\alpha \to G$ is a continuous map. At that one may provide that Z_α remains an H_α-space of purely continuous type. We set $L_{1,1} = A(s_1)$. Construct f_1 and $L_{1,q}$ by induction. Describe the q-th step.

Let $k_q = \min\{k\colon s_k \notin \bigcup_{i=1}^{q-1} L_{1,i}\}$. Suppose $s_{k_q}^2 = \widetilde{h}_{k_q}s_{k_q}^1$. Choose a clopen neighborhood $A(s_{k_q}^1)$ of $s_{k_q}^1$ with

$$A(s_{k_q}^1) \subset B_{k_q}^1 \cap \left(Z_\alpha\setminus\left(\bigcup_{i=1}^{q-1} L_{1,i}\right)\right), s_{k_q}^2 \notin A(s_{k_q}^1), \text{ and } A(s_{k_q}^1)\cap\widetilde{h}_{k_q}A(s_{k_q}^1) = \emptyset.$$

Use the same argument as above to find a clopen $L_{1,q} \subset A(s_{k_q}^1)$ and a homeomorphism $\delta_{1,q} = {\delta_{1,q}}^{-1} \in \operatorname{Int}\mathcal{H}_\alpha$ with the following properties: $L_{1,q} \supset s_{k_q}^1$, $L_{1,q}\cap S_\alpha = \widetilde{\mathcal{H}}_\alpha[L_{1,q}]\cap S_\alpha$, $\widetilde{\mathcal{H}}_\alpha[L_{1,q}]$ is clopen in Z_α, $\delta_{1,q}L_{1,q} = \widetilde{\mathcal{H}}_\alpha[L_{1,q}]\setminus L_{1,q}$, $\delta_{1,q} = \mathrm{id}$ outside of $\widetilde{\mathcal{H}}_\alpha[L_{1,q}]$, for each $s \in L_{1,q}\cap S_\alpha$

$$\lambda_\alpha(\delta_{1,q}z, z) = \xi_{1,q}(s)$$

for all $z \in L_{1,q} \cap \widetilde{\mathcal{H}}_\alpha[s]$ and some continuous map $\xi_{1,q}\colon L_{1,q} \cap S_\alpha \to G$. Note that at each step Z_α remains a purely continuous H_α-space, and S_α remains an $\widetilde{\mathcal{H}}_\alpha$-transversal on Z_α.

One has $\bigcup_{q=1}^\infty L_{1,q} \supset S_\alpha^d$. Define

$$K^1(0) = \bigcup_{q=1}^\infty L_{1,q}, \quad K^1(1) = \bigcup_{q=1}^\infty (\widetilde{\mathcal{H}}_\alpha[L_{1,q}] \setminus L_{1,q}),$$

$f_1 z = \delta_{1,q} z$ for $z \in \widetilde{\mathcal{H}}_\alpha[L_{1,q}]$, $\xi_1 s = \xi_{1,q} s$ for $s \in L_{1,q} \cap S_\alpha$. Note that $\widetilde{\mathcal{H}}_\alpha[S_\alpha \setminus (\bigcup_{i=1}^\infty L_{1,i})]$ is meager F_σ in Z_α and apply 2.5 to finish the first part of our inductive construction for λ_α.

Using the same arguments as above, the fact that $\lambda_\beta^{\hat{s}}$ is ergodic, and Lemma 2.9, construct the families $\{\widehat{L}^{1,q}\}_{q=1}^\infty$, $\{\widehat{\delta}_{1,q}\}_{q=1}^\infty \subset \operatorname{Int} \mathcal{H}_\beta$ with

$$\lambda_\beta(\widehat{\delta}_{1,q}\widehat{z}, \widehat{z}) = (\xi_1 \circ \rho)(\phi_\beta(\widehat{z}))$$

for all $\widehat{z} \in \widetilde{\mathcal{H}}_\beta[\widehat{L}_{1,q}]$. Define $\widehat{K}^1(0) = \bigcup_{q=1}^\infty \widehat{L}_{1,q} (\supset \widehat{S}_\alpha^d)$, $\widehat{K}^1(1) = \bigcup_{q=1}^\infty (\widetilde{\mathcal{H}}_\beta[\widehat{L}_{1,q}] \setminus \widehat{L}_{1,q})$, $\widehat{f}_1 z = \widehat{\delta}_{1,q} z$ for $z \in \widetilde{\mathcal{H}}_\beta[\widehat{L}_{1,q}]$, and apply 2.5. By our construction

$$\lambda_\beta(\widehat{f}_1\widehat{z}, \widehat{z}) = (\xi_1 \circ \rho)(\phi_\beta(\widehat{z})) = \lambda_\alpha(f_1 z, z)$$

for all $(z, \widehat{z}) \in K^1(0) \times \widehat{K}^1(0)$ with $\phi_\alpha(z) = \rho(\phi_\beta(\widehat{z}))$.

Describe the $(n+1)$-th step. Denote $f(\underline{\mu}) = f_1^{\mu_1} \cdot \ldots \cdot f_n^{\mu_n}$, $\widehat{f}(\underline{\mu}) = \widehat{f}_1^{\mu_1} \cdot \ldots \cdot \widehat{f}_n^{\mu_n}$. Suppose n is odd.

Suppose $\tau_{n+1} \in K^n(\underline{\mu})$. If $\underline{\mu} = \underline{0}$ let $\eta' = \tau_{n+1}$, else let $\eta' = f(\underline{\mu})\tau_{n+1}$. If $\eta' \in S_\alpha^d$ let $\eta = \tau_i$, where τ_i is an arbitrary element of $\Upsilon \cap K^n(\underline{0}) \setminus \{\tau_1\}$, otherwise $\eta = \eta'$.

Suppose $\eta = \widetilde{h}^{(m)} s_m^1$ ($h^{(m)} \in H_\alpha$). Similarly as at the first step, find a clopen neighborhood $L_{n+1,1}$ of s_m^1 with the following properties:

$$\begin{aligned}
&L_{n+1,1} \subset K^n(\underline{0}) \cap B_m^{n+1},\\
&\widetilde{h}^{(m)} L_{n+1,1} \subset K^n(\underline{0}),\\
&L_{n+1,1} \cap \widetilde{h}^{(m)} L_{n+1,1} = \emptyset,\\
&\widetilde{\mathcal{H}}_\alpha[L_{n+1,1}] \cap S_\alpha = L_{n+1,1} \cap S_\alpha,\\
&\widetilde{\mathcal{H}}_\alpha[L_{n+1,1}] \text{ is clopen.}
\end{aligned}$$

Consider $B = (\widetilde{\mathcal{H}}_\alpha[L_{n+1,1}] \cap K^n(\underline{0})) \setminus L_{n+1,1}$. Then $\phi_\alpha(B) = \phi_\alpha(L_{n+1,1})$. Apply 2.9 to the sets B and $L_{n+1,1}$ to find a homeomorphism $\delta_{n+1,1} \in \operatorname{Int} \mathcal{H}_\alpha$

with: $\delta_{n+1,1} = \delta_{n+1,1}^{-1}$, $\delta_{n+1,1}L_{n+1,1} = B$, $\delta_{n+1,1}s_m^1 = \eta$, $\delta_{n+1,1} = \mathrm{id}$ on $Z_\alpha \setminus (\widetilde{\mathcal{H}}_\alpha[L_{n+1,1}] \cap K^n(\underline{0}))$, for each $s \in L_{n+1,1} \cap S_\alpha$

$$\lambda_\alpha(\delta_{n+1,1}z, z) = \xi_{n+1,1}(s)$$

for all $z \in L_{n+1,1} \cap \widetilde{\mathcal{H}}_\alpha[s]$, where $\xi_{n+1,1}\colon L_{n+1,1} \cap S_\alpha \to G$ is a continuous map.

Proceed in the same way as at the first step to construct the families $\{L_{n+1,q}\}_{q=2}^\infty$, $\{\delta_{n+1,q}\}_{q=2}^\infty$, $\{\xi_{n+1,q}\}_{q=2}^\infty$ with the following properties: every $L_{n+1,q}$ is taken within $K^n(\underline{0})$, $\mathrm{diam}(L_{n+1,q}) < 1/(n+1)$, the homeomorphism $\delta_{n+1,q}$ interchanges $L_{n+1,q}$ with $(\widetilde{\mathcal{H}}_\alpha[L_{n+1,q}] \cap K^n(\underline{0})) \setminus L_{n+1,q}$ and is identical outside of $\widetilde{\mathcal{H}}_\alpha[L_{n+1,q}] \cap K^n(\underline{0})$. Define:

$$\begin{aligned}
&K^{n+1}(\underline{0}) = \bigcup_{q=1}^{\infty} L_{n+1,q}(\supset S_\alpha^d),\\
&K^{n+1}(\underline{0},1) = \bigcup_{q=1}^{\infty}((\widetilde{\mathcal{H}}_\alpha[L_{n+1,q}] \cap K^n(\underline{0})) \setminus L_{n+1,q}),\\
&f_{n+1}z = \delta_{n+1,q}z \quad \text{for } z \in \widetilde{\mathcal{H}}_\alpha[L_{n+1,q}] \cap K^n(\underline{0}),\\
&f_{n+1}z = f(\underline{\mu})f_{n+1}f(\underline{\mu})z \quad \text{for } z \in K^n(\underline{\mu}), \underline{\mu} \neq \underline{0},\\
&\xi_{n+1}s = \xi_{n+1,q}s \quad \text{for } s \in L_{n+1,q} \cap S_\alpha,\\
&K^{n+1}(\underline{\mu},0) = f(\underline{\mu})K^{n+1}(\underline{0}),\\
&K^{n+1}(\underline{\mu},1) = f(\underline{\mu})K^{n+1}(\underline{0},1)(\underline{\mu} \in \{0,1\}^n).
\end{aligned}$$

After that, construct $\{\widehat{L}_{n+1,q}\}_{q=1}^\infty$, $\{\widehat{K}^{n+1}(\underline{\mu})\}$, $\widehat{f}_{n+1}$ similarly, but with

$$\lambda_\beta(\widehat{f}_{n+1}\widehat{z}, \widehat{z}) = (\xi_{n+1} \circ \rho)(\phi_\beta(\widehat{z}))$$

for all $\widehat{z} \in K^{n+1}(\underline{0})$.

For n even, the construction is the same but starting with the consideration of the point $\widehat{\tau}_{n+1}$. Thus, the condition 6 also holds. ∎

Proof of Proposition 2.4: From now on let $\mathcal{K}$ stand for the space $\{0,1\}^{\mathbb{N}}$. For $\underline{\nu} \in \bigoplus_{\mathbb{N}} \mathbb{Z}_2$ denote by $C^n(\underline{\nu})$ the set $\{\underline{a} \in \mathcal{K}\colon a_1 = \nu_1, \ldots, a_n = \nu_n\}$ (the cylinder of length n), and by $|\underline{\nu}|$ — $\max\{n\colon \nu_n \neq 0\}$.

Denote by F (resp. $\widehat{F}$) the group generated by $\{f(\underline{\mu})\colon \underline{\mu} \in \{0,1\}^n, n \in \mathbb{N}\}$ (resp. $\{\widehat{f}(\underline{\mu})\colon \underline{\mu} \in \{0,1\}^n, n \in \mathbb{N}\}$). It follows from 2.8 that $H_\alpha[S_\alpha^d] = F[S_\alpha^d]$. Apply Lemma 2.6 to the groups H_α, F, and to the set $T = H_\alpha[S_\alpha^d]$. Then, without loss of generality, one may assume that H_α and F are strongly orbit equivalent on the purely continuous space Z_α. The same is true for the groups H_β and $\widehat{F}$ (on Z_β).

Note that $K^n(\underline{0}) \subset S(1/n)$, where $S(\varepsilon)$ denotes the ε-neighborhood of the closed set S_α (in the same metric on Z_α as above). It follows that $S_\alpha = \bigcap_{n=1}^\infty K^n(\underline{0})$. Hence for every $\underline{\mu} \in \{0,1\}^n$ one has

$$f(\underline{\mu})S_\alpha = \bigcap_{m=n+1}^{\infty} K^m(\underline{\mu}, 0, \ldots, 0). \tag{1}$$

For every $z \in Z_\alpha$ there exists a uniquely determined sequence $\{\underline{\mu}^n\}_{n=1}^\infty$ ($\underline{\mu}^n \in \{0,1\}^n$) such that $z \in \bigcap_{n=1}^\infty K^n(\underline{\mu}^n)$. This implies that the map $\Theta_\alpha : Z_\alpha \to S_\alpha \times \mathcal{K}$

$$\Theta_\alpha(z) = (\phi_\alpha(z), (\mu_1, \ldots, \mu_n, \ldots)),$$

where $(\mu_1, \ldots, \mu_n) = \underline{\mu}^n$, is well-defined.

One easily checks that Θ_α is continuous. We claim that Θ_α is a homeomorphism from $H_\alpha[S_\alpha]$ (with the relative topology) onto $S_\alpha \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2$ (with the relative topology inherited from $S_\alpha \times \mathcal{K}$). The bijectivity of Θ_α easily follows from $H_\alpha[S_\alpha] = F[S_\alpha]$. It remains to show the continuity of $\Theta_\alpha^{-1}: S_\alpha \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2 \to H_\alpha[S_\alpha]$.

Suppose that $(r_j, \underline{\mu}^j) \to (r, \underline{\mu})$ $(j \to \infty)$, where $r_j, r \in S_\alpha$, $\underline{\mu}^j, \underline{\mu} \in \bigoplus_{\mathbb{N}} \mathbb{Z}_2$. For an arbitrary $m \geq |\underline{\mu}|$ consider a cylinder $C^m = C^m(\underline{\mu}, 0, \ldots, 0)$ of length m. Then there exists $N_1 \in \mathbb{N}$ with $\mu^j \in C^m$ for all $j \geq N_1$. Since $r_j \in K^m(\underline{0})$ for all j, one has $f(\underline{\mu})f(\underline{\mu}^j)r_j \in K^m(\underline{0})$ for all $j \geq N_1$. As $K^m(\underline{0}) = \bigcup_{q=1}^\infty L_{m,q}$, $r \in L_{m,q_0}$ for some q_0. Hence there exists $N_2 > N_1$ with $r_j \in L_{m,q_0}$ for all $j \geq N_2$. We claim that $f(\underline{\mu})f(\underline{\mu}^j)r_j \in L_{m,q_0}$ for all $j \geq N_2$. Indeed, otherwise $f(\underline{\mu})f(\underline{\mu}^j)r_j \in L_{m,q_1}$ and then $\phi_\alpha(f(\underline{\mu})f(\underline{\mu}^j)r_j) = r_j$, so $r_j \in \widetilde{\mathcal{H}}_\alpha[L_{m,q_1}]$. But $L_{m,q_1} \cap S_\alpha = \widetilde{\mathcal{H}}_\alpha[L_{m,q_1}] \cap S_\alpha$, so $r_j \in L_{m,q_1}$ that is impossible ($L_{m,q_0} \cap L_{m,q_1} = \emptyset$). Hence for all $j \geq N_2$ and for some $s_k \in S_\alpha^d$ we have $f(\underline{\mu})f(\underline{\mu}^j)r_j \in B(s_k, 1/m) \subset B(r, 2/m)$. This proves that $f(\underline{\mu})f(\underline{\mu}^j)r_j \to r$ $(j \to \infty)$, so $f(\underline{\mu}^j)r_j \to f(\underline{\mu})r$ $(j \to \infty)$.

Apply Lavrentiev's Theorem ([14]) to find an extension of Θ_α to a homeomorphism of two G_δ-sets: Θ_α: $Z_0 \to K_0$, where $H_\alpha[S_\alpha] \subset Z_0 \subset Z_\alpha$, $S_\alpha \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2 \subset K_0 \subset S_\alpha \times \mathcal{K}$.

Let us consider a dense H_α-invariant G_δ-subset of Z_α: $Z_1 = Z_\alpha \setminus (H_\alpha[Z_\alpha \setminus Z_0])$. It is a purely continuous H_α-space containing S_α as a $\widetilde{\mathcal{H}}_\alpha$-transversal. Let $K_1 = \Theta_\alpha(Z_1)$; this is a dense G_δ-subset of $S_\alpha \times \mathcal{K}$. From now on we consider Θ_α as a homeomorphism from Z_1 onto K_1.

Denote by $a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)$ the action of $\bigoplus_{\mathbb{N}} \mathbb{Z}_2$ on the second coordinate of $S_\alpha \times \mathcal{K}$:

$$a^{(2)}(\underline{\mu})(s, \underline{\nu}) = (s, \underline{\mu} + \underline{\nu}).$$

For every $\underline{\mu}, \underline{\nu} \in \bigoplus_{\mathbb{N}} \mathbb{Z}_2$ one has $\Theta_\alpha f(\underline{\mu})\Theta_\alpha^{-1} = a^{(2)}(\underline{\mu})(s, \underline{\nu})$. It follows that K_1 is $\bigoplus_{\mathbb{N}} \mathbb{Z}_2$-invariant, and $a^{(2)}(\underline{\mu})|_{K_1} \equiv \Theta_\alpha f(\underline{\mu})\Theta_\alpha^{-1}$. Hence the groups $\Theta_\alpha H_\alpha \Theta_\alpha^{-1}$ and $a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)$ have the same orbits on K_1. It should be also noted that $\Theta_\alpha(\widetilde{\mathcal{H}}_\alpha[s]) = (\{s\} \times \mathcal{K}) \cap K_1$.

Similarly construct a homeomorphism Θ_β: $\widehat{Z}_1 \to \widehat{K}_1$, having the same properties as Θ_α, where $H_\beta[S_\beta] \subset \widehat{Z}_1 \subset Z_\beta$, $S_\beta \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2 \subset \widehat{K}_1 \subset S_\beta \times \mathcal{K}$. One may assume without loss of generality that $\widehat{K}_1 = K_1$. Let

$$\Theta = \Theta_\alpha^{-1} \circ (\rho \times \mathrm{id}) \circ \Theta_\beta\colon \widehat{Z}_1 \to Z_1.$$

Then $\Theta \times \Theta$: $\mathcal{H}_\alpha \to \mathcal{H}_\beta$ is an isomorphism, and the cocycles λ_α and $\lambda_\beta \circ (\Theta \times \Theta)$ coincide by 2.8, i.8. ∎

Remark: One can observe from the proof of Proposition 2.4 that any non-ergodic purely continuous action of a countable group is orbit equivalent to the natural action of $\bigoplus_{\mathbb{N}} \mathbb{Z}_2$ on $Y \times \{0,1\}^{\mathbb{N}}$, where Y is a Polish space. This fact, together with [6, prop. 8,10] and [19, Th. 1.8], gives us a complete generic orbit classification of countable homeomorphism groups of a Polish space (in particular, it implies that an equivalence relation generated by an arbitrary countable homeomorphism group of a Polish space is generically hyperfinite (cf. [12, 12.1])).

Now turn to a proof of weak equivalence of φ_α and φ_β.

It follows from the proof of the previous proposition that one may assume $S_\alpha = S_\beta = S$, $Z_\alpha = Z_\beta = S \times \mathcal{K} = Z$ modulo meager sets, and $W_\alpha(G) = W_\beta(G) = W(G)$ is an action on S. Besides, $\mathcal{H}_\alpha = \mathcal{H}_\beta = \mathcal{R}_{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)} \subset (S \times \mathcal{K}) \times (S \times \mathcal{K})$, $\lambda_\alpha = \lambda_\beta = \lambda$ on $\mathcal{R}_{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)}$.

Let

$$V_\alpha(G) = \Theta_\alpha V(G) {\Theta_\alpha}^{-1}, \quad V_\beta(G) = ((\rho \times \mathrm{id}) \circ \Theta_\beta) V(G) ((\rho \times \mathrm{id}) \circ \Theta_\beta)^{-1},$$
$$E_\alpha(\Gamma) = \Theta_\alpha(\Gamma(\alpha)){\Theta_\alpha}^{-1}, \quad E_\beta(\Gamma) = ((\rho \times \mathrm{id}) \circ \Theta_\beta)(\Gamma(\beta))((\rho \times \mathrm{id}) \circ \Theta_\beta)^{-1}.$$

Note that $V_\alpha(G)$ and $V_\beta(G)$ normalize the action $a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)$ (i.e. $V_\alpha(G)$, $V_\beta(G) \subset \operatorname{Aut} \mathcal{R}_{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)}$). Transfer the cocycle φ_α (resp. φ_β) by Θ_α (resp. by $(\rho \times \mathrm{id}) \circ \Theta_\beta$) onto the equivalence relation on $S \times \mathcal{K}$ generated by $a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)$ and $V_\alpha(G)$ (resp. $V_\beta(G)$). We keep the notation φ_α for the cocycle

$$\varphi_\alpha \circ ({\Theta_\alpha}^{-1} \times {\Theta_\alpha}^{-1}) \in Z^1(\mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}}, G).$$

The same agreement applies to φ_β. As above, ϕ denotes the projection $S \times \mathcal{K} \to S$.

We need the following lemma formulated in a more general context.

LEMMA 2.10: *Let $W(G)$ be an ergodic action of a countable group G by homeomorphisms of a perfect Polish space S. Then there exists a cocycle $c \in Z^1(\mathcal{R}_{W(G)}, G)$ such that $W(c(s_2, s_1))s_1 = s_2$ for all $(s_1, s_2) \in \mathcal{R}_{W(G)}$.*

Proof: By virtue of [19, 1.8], there exists an ergodic homeomorphism θ of S generating the equivalence relations $\mathcal{R}_{W(G)}$. Let $\sigma_1, \sigma_2 \colon G \times S \to S$ be (Borel) maps defined by: $\sigma_1(g, s) = W(g)s$, $\sigma_2(g, s) = \theta s$. Then the set

$$A = \{(g, s) \in G \times S \colon \sigma_1(g, s) = \sigma_2(g, s)\}$$

is Borel, and $\pi(A) = S$, where π is the projection $G \times S \to S$. Therefore there exists a Borel $B \subset G \times S$ such that $\pi \colon B \to S$ is a bijection ([11]). Let $B_g = \pi(B \cap (\{g\} \times S))$. Define a map $\xi \colon S \to G$ by $\xi(s) = g$ if $s \in B_g$. Then the cocycle $c = c(\xi) \in Z^1(\mathcal{R}_{\mathbb{Z}(\theta)}, G)$ constructed by means of ξ (see Section 1) satisfies the conditions of the lemma. ∎

LEMMA 2.11: *Let A, B be clopen subsets of Z with $\phi(A) \cap \phi(B) = \emptyset$, $\{s_k\}_{k=1}^{\infty}$ a dense subset of S. Let $\delta \in \operatorname{Int} \mathcal{R}_{W(G)}$ be a homeomorphism of S with $\delta = \delta^{-1}$, $\delta = \mathrm{id}$ outside of $\phi(A) \cup \phi(B)$, and $\delta\phi(A) = \phi(B)$. Then there exists $\widetilde{\delta} \in \operatorname{Int} \mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}}$ with the following properties: $\widetilde{\delta} = \widetilde{\delta}^{-1}$, $\widetilde{\delta} = \mathrm{id}$ outside of $A \cup B$, $\widetilde{\delta}A = B$, for each $z \in Z$:*

$$\widetilde{\delta} z = V_\alpha(c(\delta\phi(z), \phi(z)))E_\alpha(\gamma)z$$

for some $\gamma (= \gamma(z)) \in \Gamma$, where $c \in Z^1(\mathcal{R}_{W(G)}, G)$ is a cocycle from Lemma 2.10. If then some meager set is discarded from Z, one may provide every set $\{s_k\} \times \mathcal{K}$ remains comeager in $\mathcal{K}$.

Proof: Let $\{A_m\}_{m \le \mathbb{N}}$ be a disjoint family of clopen subsets of S with $\bigcup_m A_m = \phi(A)$, $\delta s = W(g_m)s$ for all $s \in A_m$ and some $g_m \in G$. One may assume that every set $A_{m,k} = \{s \in A_m \colon c(W(g_m)s, s) = g_{m,k}\}$ is clopen. It suffices for every m, k to interchange $(\phi(A_{m,k}) \times \mathcal{K}) \cap A$ and $(\phi(W(g_{m,k})A_{m,k}) \times \mathcal{K}) \cap B$ using elements of the group $E_\alpha(\Gamma)$ and the homeomorphism $V_\alpha(g_{m,k})$. It can be done by the same methods as in the proof of Lemma 2.9. ∎

It will be convenient also to identify S with $\mathcal{K}$, on which the natural action of $\bigoplus_{\mathbb{N}} \mathbb{Z}_2$, say $u(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)$, generates the same equivalence relation as $W(G)$ (see [19, 1.7]). Thus one has $Z = \mathcal{K} \times \mathcal{K}$ modulo meager sets.

Fix a (dense) orbit $W(G)s_0$, ($s_0 \in S$). One may assume without loss of generality that by the identification of S with $\mathcal{K}$, $s_0 = 0$ and $W(G)s_0 = \{s_{\underline{\mu}}\}_{\underline{\mu} \in \bigoplus_{\mathbb{N}} \mathbb{Z}_2}$, where $s_{\underline{\mu}} = u(\underline{\mu})s_0$ ([19,1.7, 1.8]). Let $t_0 = (s_0, 0) \in S \times \mathcal{K}$.

There exists $t_1 \in C^1(1) \times C^1(0)$ with $(t_1, V_\alpha(c(s_1, s_0))t_0) \in \mathcal{R}_{E_\alpha(\Gamma)}$ (here and below c is a cocycle from Lemma 2.10). Similarly, there exists $\widehat{t_1} \in C^1(1) \times C^1(0)$ with $(\widehat{t_1}, V_\beta(c(s_1, s_0))t_0) \in \mathcal{R}_{E_\beta(\Gamma)}$.

Use 2.11 to find a homeomorphism $\widetilde{p}(1) \in \operatorname{Int} \mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}}$ with the following properties: $\widetilde{p}(1)^{-1} = \widetilde{p}(1)$, $\widetilde{p}(1)t_0 = t_1$, $\widetilde{p}(1)(C^1(0) \times \mathcal{K}) = C^1(1) \times \mathcal{K}$, and for every $(s,k) \in S \times \mathcal{K}$

$$\widetilde{p}(1)(s,k) = V_\alpha(c(u(1)s, s))E_\alpha(\gamma)(s,k)$$

for some $\gamma = \gamma(s,k) \in \Gamma$.

Define:

$$\begin{gathered} M^1(0;0) = C^1(0) \times C^1(0), \quad M^1(1;0) = \widetilde{p}(1)M^1(0;0), \\ M^1(0;1) = C^1(0) \times C^1(1), \quad M^1(1;1) = \widetilde{p}(1)M^1(0;1), \\ p(0;0) = \mathrm{id}, \\ p(1;0) = \widetilde{p}(1), \\ p(0;1)z = a^{(2)}(1)z \quad \text{for } z \in C^1(0) \times \mathcal{K}, \\ p(0;1)z = \widetilde{p}(1)a^{(2)}(1)\widetilde{p}(1)z \quad \text{for } z \in C^1(1) \times \mathcal{K}, \\ p(1;1) = p(0;1)p(1;0). \end{gathered}$$

Construct $\{\widehat{p}(j_1; j_2), \widehat{M}^1(j_1; j_2)\colon j_1, j_2 \in \{0,1\}\}$ associated to β in the same way.

At the n-th step of the inductive construction, consider $s_{\underline{i}}$, where $\underline{i} = (0, \ldots, 0, 1)$, $|\underline{i}| = n$, and find $t_{\underline{i}} \in C^{n-1}(\underline{0}) \times C^{n-1}(\underline{0})$ with $(t_{\underline{i}}, V_\alpha(c(s_{\underline{i}}, s_0))t_0) \in \mathcal{R}_{E_\alpha(\Gamma)}$. By virtue of 2.11 there exists a homeomorphism $\widetilde{p}(n) \in \operatorname{Int} \mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}}$ which has the following properties: $\widetilde{p}(n)^{-1} = \widetilde{p}(n)$, $\widetilde{p}(n) = \mathrm{id}$ outside of $C^{n-1}(\underline{0}) \times C^{n-1}(\underline{0})$, $\widetilde{p}(n)t_0 = t_{\underline{i}}$, $\widetilde{p}(n)(C^n(\underline{0}) \times C^{n-1}(\underline{0})) = C^n(\underline{i}) \times C^{n-1}(\underline{0})$, and for every $(s,k) \in C^{n-1}(\underline{0}) \times C^{n-1}(\underline{0})$

$$\widetilde{p}(n)(s,k) = V_\alpha(c(u(\underline{i})s, s))E_\alpha(\gamma)(s,k)$$

for some $\gamma = \gamma(s,k) \in \Gamma$.

Define

$$\begin{gathered} M^n(\underline{0};\underline{0}) = C^n(\underline{0}) \times C^n(\underline{0}), \quad M^n(\underline{i};\underline{0}) = \widetilde{p}(n)M^n(\underline{0};\underline{0}), \\ M^n(\underline{0};\underline{i}) = C^n(\underline{0}) \times C^n(\underline{i}), \quad M^n(\underline{i};\underline{i}) = \widetilde{p}(n)M^n(\underline{0};\underline{i}), \end{gathered}$$

$$M^n((\underline{\mu},j_1);(\underline{\nu},j_2)) = p(\underline{\mu};\underline{\nu})M^n((\underline{0},j_1);(\underline{0},j_2)),$$
$$p(\underline{0};\underline{0}) = \mathrm{id},$$
$$p(\underline{i};\underline{0})z = \widetilde{p}(n)z \quad \text{for } z \in M^{n-1}(\underline{0};\underline{0}),$$
$$p(\underline{0};\underline{i})z = a^{(2)}(\underline{i})z \quad \text{for } z \in C^n(\underline{0}) \times C^{n-1}(\underline{0}),$$
$$p(\underline{0};\underline{i})z = \widetilde{p}(n)a^{(2)}(\underline{i})\widetilde{p}(n)z \quad \text{for } z \in C^n(\underline{i}) \times C^{n-1}(\underline{0}),$$
$$p((\underline{\mu},j_1);(\underline{\nu},j_2))z = p(\underline{\mu};\underline{\nu})p((\underline{0},j_1);\underline{0})p(\underline{0};(\underline{0},j_2))p(\underline{\mu};\underline{\nu})z \;\text{ for } z \in M^{n-1}(\underline{\mu};\underline{\nu}),$$

where $\underline{\mu},\underline{\nu} \in \{0,1\}^{n-1}, j_1, j_2 \in \{0,1\}$.

It should be noted that $p(\underline{\mu};\underline{0})(\{s_0\} \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2) = a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)[t_{\underline{\mu}}]$ because $V_\alpha(G)$ normalizes the action $a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2)$.

Thus, at the n-th step we construct the clopen partition of Z: $\{M^n(\underline{\mu};\underline{\nu})\colon \underline{\mu},\underline{\nu} \in \{0,1\}^n\}$ and the family of pairwise commuting homeomorphisms $\{p(\underline{\mu};\underline{\nu})\colon \underline{\mu},\underline{\nu} \in \{0,1\}^n\} \subset \mathrm{Int}\,\mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}}\mathbb{Z}_2),V_\alpha(G)\}}$ with $p(\underline{\mu};\underline{\nu}) = p(\underline{\mu};\underline{\nu})^{-1}$. Besides, one has $p(\underline{\mu};\underline{\nu})M^n(\underline{0};\underline{0}) = M^n(\underline{\mu};\underline{\nu})$. Construct similarly the same objects $\{\widehat{p}(\underline{\mu};\underline{\nu}), \widehat{M}^n(\underline{\mu};\underline{\nu})\colon \underline{\mu},\underline{\nu} \in \{0,1\}^n\}$ associated to β.

It is not difficult to check that

$$\begin{aligned} p(\underline{\mu};\underline{\nu}) &= \prod_{k,j} p((0,\dots,\mu_k,\dots,0);(0,\dots,\nu_j,\dots,0)) \\ &= \prod_{k,j} p((0,\dots,\mu_k);(0,\dots,\nu_j)) \end{aligned}$$

for all $\underline{\mu},\underline{\nu} \in \{0,1\}^n$, $n \in \mathbb{N}$. So one may assume that $p(\cdot;\cdot)$ is well-defined on $\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2$, and, moreover, this is a $\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2$-action on Z by homeomorphisms. Observe that $p(\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2)[t_0] = \{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}[t_0]$. Then 2.6 (or [19, 1.5]) implies that the groups $p(\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2)$ and $\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}$ have the same orbits on Z.

Consider the sets $p(\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2)[t_0]$ and $\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2$ with the relative topologies inherited from Z and $\mathcal{K} \times \mathcal{K}$ respectively. Define a map

$$\overline{\Theta}_\alpha\colon p\Big(\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2\Big)[t_0] \to \bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2$$

by

$$\overline{\Theta}_\alpha(p(\underline{\mu};\underline{\nu})t_0) = (\underline{\mu},\underline{\nu}).$$

Since $t_0 = \bigcap_{n=1}^{\infty} M^n(\underline{0};\underline{0})$ one has

$$p(\underline{\mu};\underline{\nu})t_0 = \bigcap_{n=\max\{|\underline{\mu}|,|\underline{\nu}|\}+1}^{\infty} M^n((\underline{\mu},0,\dots,0);(\underline{\nu},0,\dots,0))$$

Then by the standard arguments (see [19, 1.8] or Prop. 2.4 above) $\overline{\Theta}_\alpha$ is a homeomorphism. By Lavrentiev's Theorem ([14]), there exists an extension of $\overline{\Theta}_\alpha$ to a homeomorphism of G_δ-sets: $\overline{\Theta}_\alpha\colon Z_1 \to K_1$, where $p(\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2)[t_0] \subset Z_1$, $\bigoplus_{\mathbb{N}} \mathbb{Z}_2 \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2 \subset K_1$. Analogously, construct $\overline{\Theta}_\beta$ having the same properties as $\overline{\Theta}_\alpha$. Arguing as in the proof of 2.4, we conclude that

$$\overline{\Theta} = \overline{\Theta}_\alpha^{\,-1} \circ \overline{\Theta}_\beta\colon \mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\beta(G)\}} \to \mathcal{R}_{\{a^{(2)}(\bigoplus_{\mathbb{N}} \mathbb{Z}_2), V_\alpha(G)\}}$$

is an isomorphism.

Observe that, by our construction, $\varphi_\alpha(p(\underline{\mu};\underline{0})z, z) = c(u(\underline{\mu})\phi(z), \phi(z))$. Therefore, if $\phi(z) = \phi(\widehat{z})$, then

$$\varphi_\alpha(p(\underline{\mu};\underline{0})z, z) = \varphi_\beta(\widehat{p}(\underline{\mu};\underline{0})\widehat{z}, \widehat{z}).$$

Besides,

$$\varphi_\alpha(p(\underline{0};\underline{\mu})z, z) = \lambda(a^{(2)}(\underline{\mu})z, z) = \varphi_\beta(\widehat{p}(\underline{0};\underline{\mu})\widehat{z}, \widehat{z})$$

for all $(z, \widehat{z}) \in M^n(\underline{0};\underline{0}) \times M^n(\underline{0};\underline{0}) (= \widehat{M}^n(\underline{0};\underline{0}) \times \widehat{M}^n(\underline{0};\underline{0}))$ with $\phi(z) = \phi(\widehat{z})$, $\underline{\mu} \in \{0,1\}^n$. It follows that

$$\varphi_\alpha(p(\underline{\mu};\underline{\nu})z, z) = \varphi_\beta(\widehat{p}(\underline{\mu};\underline{\nu})\widehat{z}, \widehat{z})$$

for all $\underline{\mu}, \underline{\nu} \in \bigoplus_{\mathbb{N}} \mathbb{Z}_2$ and all $(z, \widehat{z}) \in M^n(\underline{0};\underline{0}) \times M^n(\underline{0};\underline{0})$ with $\phi(z) = \phi(\widehat{z})$ and $n = \max\{|\underline{\mu}|, |\underline{\nu}|\}$. This implies

$$\varphi_\alpha \circ (\overline{\Theta}_\alpha^{\,-1} \times \overline{\Theta}_\alpha^{\,-1}) = \varphi_\beta \circ (\overline{\Theta}_\beta^{\,-1} \times \overline{\Theta}_\beta^{\,-1}),$$

so φ_α and φ_β are weakly equivalent. ∎

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